LECTURE 4: BEILINSON'S CONJECTURES FOR NUMBER FIELDS
(THE WORK OF BOREL AND APPLICATIONS)

In this lecture we shall study Beilinson's conjectures in the special case that $X = \operatorname{Spec} F$ for F a number field. That is, the dimension zero case. This is the only dimension in which Beilinson's conjectures are known in generality. In some sense, the proof (by Borel) inspired Bloch, and in turn Beilinson, in their hunt for a higher dimensional generalisation. I hope to convey at least glimpses of what I believe is a truly beautiful story.

1. Motivic cohomology of a number field

Let us see what Beilinson's conjectures predict for the Dedekind zeta function $\zeta_F(s) = L(H^0(F), s)$ of a number field F. Let $[F : \mathbb{Q}] = d = r_1 + 2r_2$ as usual. Recall that $\zeta_F(s)$ has a single (simple) pole, at $s = 1$. The Euler factors at infinity contribute r_1 copies of $\Gamma(\frac{s}{2})$ and r_2 copies of $\Gamma(s)$. Since the Gamma function has simple poles at $s = 0, -1, -2, \ldots$ one reads off the from functional equation $s \leftrightsquigarrow 1 - s$ that the order of vanishing of $\zeta_F(s)$ at $s = -n$ is precisely d_n where

$$d_n := \begin{cases} r_1 + r_2 - 1 & \text{if } n = 0 \\ r_2 & \text{if } n > 0 \text{ is odd} \\ r_1 + r_2 & \text{if } n > 0 \text{ is even}. \end{cases}$$

Write $X = \operatorname{Spec} F$ and let $\mathcal{O}_F$ be the ring of integers of F. Let $m \geq 1$ and consider the Beilinson regulator

$$r_{\mathcal{B}} : H^1_{\mathcal{M}}(\mathcal{O}_F, \mathbb{Q}(m)) \to H^1_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(m)) \cong \mathbb{R}^{d_{m-1}}.$$

Then the Beilinson conjectures predict first that

$$\dim_{\mathbb{Q}} H^1_{\mathcal{M}}(\mathcal{O}_F, \mathbb{Q}(m)) = d_{m-1}$$

for $m \geq 1$ and second that

$$\lim_{s \to 1-m} (s-(m-1))^{-d_{m-1}} \zeta_F(s) \sim_{\mathbb{Q}^*} \begin{cases} \operatorname{vol}(r_{\mathcal{B}}(H^1_{\mathcal{M}}(\mathcal{O}_F, \mathbb{Q}(m)))) & \text{if } m > 1 \\ \operatorname{vol}(\tilde{r}_{\mathcal{B}}(H^1_{\mathcal{M}}(\mathcal{O}_F, \mathbb{Q}(m))) \oplus N^0(X)_{\mathbb{Q}}) & \text{if } m = 1. \end{cases}$$

Note that $N^0(X) = \mathbb{Z}$ in our case of $X = \operatorname{Spec} F$. Under the functional equation, for $m > 1$ the above becomes

$$|\operatorname{Disc}(F)|^{\frac{1}{2}} \pi^{m(d_m - d)} \zeta_F(m) \sim_{\mathbb{Q}^*} \operatorname{vol}(r_{\mathcal{B}}(H^1_{\mathcal{M}}(\mathcal{O}_F, \mathbb{Q}(m)))).$$

At the end of Lecture 1 I stated Borel's theorem on the dimensions of $H^1_{\mathcal{M}}(\mathcal{O}_F, \mathbb{Q}(m))$ and lo and behold they confirm the first part of Beilinson's conjectures for $X = \operatorname{Spec} F$.

Remark 1.1. Of course, I stated Borel's theorem for $H^1_{\mathcal{M}}(F, \mathbb{Q}(m))$ but $H^1_{\mathcal{M}}(\mathcal{O}_F, \mathbb{Q}(m)) \simeq H^1_{\mathcal{M}}(F, \mathbb{Q}(m))$ for $m > 1$ (by the localisation sequence and since the motivic cohomology of a finite field is torsion away from the Chow diagonal) and $H^1_{\mathcal{M}}(\mathcal{O}_F, \mathbb{Q}(m)) \cong \mathcal{O}_F^* \otimes \mathbb{Q}$.

And what about the second part of the conjecture? Well, Borel also defined a regulator map (called the Borel regulator $r_{\mathcal{B}o}$) and showed that the associated covolume agrees with the residues of zeta. Subsequent work of Burgos-Gil showed that $r_{\mathcal{B}o} = 2r_{\mathcal{B}}$, hence confirming the second part of Beilinson's conjectures for $X = \operatorname{Spec} F$.

Remark 1.2. The above story is completely ahistorical; Borel's work is from the 70's and was an inspiration for Bloch and Beilinson's work in the 80's.

To give a precise statement of Borel's theorem and to describe the proof, it will be easier to phrase things in terms of algebraic K-theory (recall that I stated in Lecture 1 that there is a motivic Atiyah-Hirzebruch spectral sequence which links algebraic K-theory and motivic cohomology).

Theorem 1.3. *(Borel) The even K-groups $K_{2n}(\mathcal{O}_F)$ are torsion. For the odd K-groups we have $\dim_{\mathbb{Q}} K_{2n+1}(\mathcal{O}_F) \otimes \mathbb{Q} = d_n$. Moreover, there is a regulator map*

$$r_{\mathcal{B}o} : K_{2n+1}(\mathcal{O}_F) \to \mathbb{R}^{d_n}$$

whose image has covolume (wrt to a natural $\mathbb{Q}$-structure on the right hand side)

$$\operatorname{vol}(r_{\mathcal{B}o}(K_{2n+1}(\mathcal{O}_F) \otimes \mathbb{Q})) \sim_{\mathbb{Q}^*} |\operatorname{Disc}(F)|^{\frac{1}{2}} \pi^{(n+1)(d_{n+1}-d)} \zeta_F(n+1)\,.$$

Remark 1.4. The torsion subgroups of $K_*(\mathcal{O}_F)$ also turn out to be very interesting, but that's a whole other story...

1.5. **A sketch proof of Borel's theorem.**
This section will barely give any details. Don't worry if you don't understand some (or any) of the objects involved. The point that I want to make is that none of these tools are available in higher dimension.

The dimension part of Borel's theorem relies on two other theorems of Borel:

Theorem 1.6. *(Borel) Let G be an algebraic group over $\mathbb{Q}$ such that $G(\mathbb{R})$ is connected, and let $\Gamma \subset G(\mathbb{Q})$ be an arithmetic subgroup. Then the map*

$$H^q_{\mathrm{cont}}(G(\mathbb{R}), \mathbb{R}) \to H^q(\Gamma, \mathbb{R}).$$

induced by $\Gamma \to G(\mathbb{R})$ is an isomorphism for $q \ll \operatorname{rank}_{\mathbb{Q}} G(\mathbb{Q})$.

On the left hand side we have continuous group cohomology, where $\mathbb{R}$ is considered as a trivial $G(\mathbb{R})$-module with the usual topology and on the right hand side the group cohomology of the discrete subgroup Γ. We will use the theorem in the case $G = \operatorname{Res}_{F/\mathbb{Q}}\mathrm{SL}_N$ and $\Gamma = \mathrm{SL}_N(\mathcal{O}_F)$.

Theorem 1.7. *(Borel)*

$$H^*_{\mathrm{cont}}(\mathrm{SL}_N(\mathbb{R}), \mathbb{R}) \cong H^*(\mathrm{SO}_N(\mathbb{R})\backslash \mathrm{SU}_N(\mathbb{R}), \mathbb{R}) \cong \wedge^*(e_5, e_9, e_{13}, \ldots, e_{4\lfloor \frac{N-1}{2} \rfloor + 1})$$

where each $e_q \in H^q(\mathrm{SO}_N(\mathbb{R})\backslash \mathrm{SU}_N(\mathbb{R}), \mathbb{Z})$ and

$$H^*_{\mathrm{cont}}(\mathrm{SL}_N(\mathbb{C}), \mathbb{R}) \cong H^*(\mathrm{SU}_N(\mathbb{C}), \mathbb{R}) \cong \wedge^*(\varepsilon_3, \varepsilon_5, \epsilon_7, \ldots, \varepsilon_{2N-1})$$

where each $\varepsilon_q \in H^q(\mathrm{SU}_N(\mathbb{C}), \mathbb{Z})$.

Now, by definition $K_m(\mathcal{O}_F) := \pi_m(\mathrm{BGL}(\mathcal{O}_F)^+)$. The short exact sequence

$$0 \to \mathrm{SL}(\mathcal{O}_F) \to \mathrm{GL}(\mathcal{O}_F) \to \mathcal{O}_F^* \to 0$$

implies that $K_m(\mathcal{O}_F) := \pi_m(\mathrm{BSL}(\mathcal{O}_F)^+)$ for $m \geq 2$. So there is a Hurewicz map $K_m(\mathcal{O}_F) \to H_m(\mathrm{BSL}(\mathcal{O}_F)^+, \mathbb{Z})$. The topological space $\mathrm{BSL}(\mathcal{O}_F)^+$ is a associative

H-space (a topological space with a continuous product map which is associative up to homotopy. The product makes the homology and cohomology into dual Hopf algebras). By Cartan-Serre, for any H-space H the Hurewicz map $\otimes\mathbb{Q}$ satisfies

$$\pi_m(H)\otimes\mathbb{Q} \xrightarrow{\sim} PH_m(H,\mathbb{Q}) := \{x \in H_m(H,\mathbb{Q}) \,|\, \Delta_* x = x\otimes 1 + 1\otimes x\}.$$

The right hand side is called the primitive homology. It is the dual of the indecomposable cohomology (i.e. H^m modulo the image of cup product). So we get an isomorphism

$$K_m(\mathcal{O}_F)\otimes\mathbb{Q} \xrightarrow{\sim} PH_m(\mathrm{BSL}(\mathcal{O}_F)^+,\mathbb{Q}) \cong PH_m(\mathrm{BSL}(\mathcal{O}_F)^+,\mathbb{Q}) \cong \varinjlim_N PH_m(\mathrm{SL}_N(\mathcal{O}_F),\mathbb{Q})\,.$$

Now let $G := \mathrm{Res}_{F/\mathbb{Q}}\mathrm{SL}_N(\mathbb{R})$ so that $G(\mathbb{R}) = \mathrm{SL}_N(\mathbb{R}\otimes_{\mathbb{Q}} F) \cong \mathrm{SL}_N(\mathbb{R})^{r_1}\times \mathrm{SL}_N(\mathbb{C})^{r_2}$. Then the above two theorems combine to give

$$H^m(\mathrm{SL}_N(\mathcal{O}_F),\mathbb{R}) \cong H^m_{\mathrm{cont}}(G(\mathbb{R}),\mathbb{R}) \cong \wedge^m(e_i)^{\otimes r_1}\otimes\wedge^m(\epsilon_j)^{\otimes r_2}\,.$$

So we see that $H^m(\mathrm{SL}_N(\mathcal{O}_F),\mathbb{R}) \cong H^m(\mathrm{SL}_{N+1}(\mathcal{O}_F),\mathbb{R})$ for $N \gg 0$ and thus

$$H^m(\mathrm{SL}(\mathcal{O}_F),\mathbb{R}) \cong \wedge^m(e_i)^{\otimes r_1}\otimes\wedge^m(\epsilon_j)^{\otimes r_2}\,.$$

Taking indecomposables and then taking duals gives the result:

$$\dim_{\mathbb{Q}}(K_m(\mathcal{O}_F)\otimes\mathbb{Q}) = \dim_{\mathbb{R}} PH_m(\mathrm{SL}(\mathcal{O}_F,\mathbb{R}) = \begin{cases} 0 & \text{if } m \text{ is even} \\ d_n & \text{if } m = 2n+1\,. \end{cases}$$

We won't talk about the proof of the residue part of Borel's theorem. But at least let us see what Borel's regulator is. Well, for each embedding $\sigma : F \hookrightarrow \mathbb{C}$ (real or complex) we can consider the composition

$$K_{2n+1}(\mathcal{O}_F) \to PH_{2n+1}(\mathrm{SL}(\mathcal{O}_F),\mathbb{Z}) \xrightarrow{\sigma_*} PH_{2n+1}(\mathrm{SL}(\mathbb{C}),\mathbb{Z}) \xrightarrow{(2\pi i)^n c_{2n+1}} \mathbb{R}(n)$$

where $c_{2n+1} := 2\pi\varphi^{-1}(\epsilon_{2n+1})$ and $\varphi : H^*_{\mathrm{cont}}(\mathrm{SL}_N(\mathbb{C}),\mathbb{R}) \xrightarrow{\sim} \wedge^*(\varepsilon_3,\varepsilon_5,\epsilon_7,\ldots,\varepsilon_{2N-1})$ is the isomorphism in Theorem 1.7. Ranging over all the embeddings we get the Borel regulator

$$r_{\mathcal{B}o} = r_{\mathcal{B}o}^{(n+1)} : K_{2n+1}(\mathcal{O}_F) \to \mathbb{R}^{d_n}\,.$$

2. An application: Zagier's conjecture

In the above formulation, it is not too hard to figure out that under the identification $\mathcal{O}_F^* \cong K_1(\mathcal{O}_F) \cong H_1(\mathrm{GL}_N(\mathcal{O}_F),\mathbb{Z})$ (the latter isomorphism holds for $N \geq 3$), the components of the Dirichlet regulator are given by $\log|\det| \in \mathrm{Hom}_{\mathrm{cont}}(\mathrm{GL}_N(\mathbb{C}),\mathbb{R}) \cong H^1_{\mathrm{cont}}(\mathrm{GL}_N(\mathbb{C}),\mathbb{R}) \cong \mathbb{R}$ for each embedding $F \hookrightarrow \mathbb{C}$. We knew this already from Lecture 1 since the way I defined Dirichlet regulator was in terms of logarithms of algebraic numbers. In any case, the $m = 1$ case of Beilinson's conjecture for $\mathrm{Spec}\, F$ (i.e. the analytic class number formula) says that up to multiplying by a non-zero rational number and a specified power of π, the value of $\zeta_F(s)/\zeta_{\mathbb{Q}}(s)$ at $s = 1$ is given as a product of logarithms of algebraic numbers in F. We cannot simply take the value at $s = 1$ of $\zeta_F(s)$ because it has a pole there. But at integers $s \geq 2$, $\zeta_F(s)$ does not have a pole so we can speak of its value. In this case, we have Zagier's conjecture which loosely says that up to a rational number and a specified power of π, the values of $\zeta_F(s)$ at integers $s = k \geq 2$ are given by k-th polylogarithms of algebraic numbers.

Recall that for $k \geq 1$, the k-th polylogarithm is defined on $\{z \in \mathbb{C} \mid |z| < 1\}$ by

$$\mathrm{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}$$

The case $k = 1$ is the usual logarithm $-\log(1-z)$, $k = 2$ is called the dilogarithm, $k = 3$ the trilogarithm, and so on. Polylogarithms crop up all over the place in maths and physics, and have been studied since at least the 17th century. In modern times they provide links/cross-fertilisation between the mixed motives and quantum field theory communities, for example!

Just like the logarithm, polylogarithms admit an analytic continuation to a multi-valued function on $\mathbb{C}\backslash\{0,1\}$. Bloch, Wigner, Ramakrishnan and Zagier showed how to define a single-valued version of the polylogarithm. For example, the single-valued form of the dilogarithm Li_2 (called the Bloch-Wigner function) is

$$D : \mathbb{C}\backslash\{0,1\} \to \mathbb{R}, \qquad z \mapsto \mathrm{Im}(\mathrm{Li}_2(z)) + \arg(1-z)\log|z|\,.$$

It satisfies the so-called Spence-Abel 5-term relation:

$$D(x) + D(y) + D\left(\frac{1-x}{1-xy}\right) + D(1-xy) + D\left(\frac{1-y}{1-xy}\right) = 0$$

for all $x, y \in \mathbb{C} \setminus \{0,1\}$ with $xy \neq 1$.

Let's look closer at the first case of Zagier's conjecture, "$\zeta_F(2)$ and dilogarithms". Since the "$\zeta_F(1)/\zeta_{\mathbb{Q}}(1)$ and logarithms" case was coming from the Dirichlet/Borel/Beilinson regulator $K_1(\mathcal{O}_F) \to \mathbb{R}^{r_1+r_2-1}$, it makes sense to look at the Borel/Beilinson regulator

$$r_{\mathcal{B}o} : K_3(\mathcal{O}_F) \otimes \mathbb{Q} \cong \mathrm{CH}^2(\mathcal{O}_F, 3) \otimes \mathbb{Q} = H^1_{\mathcal{M}}(\mathcal{O}_F, \mathbb{Q}(2)) \to \mathbb{R}^{d_1} = \mathbb{R}^{r_2}$$

to study the $s = 2$ case of Zagier's conjecture. Motivated by the Spence-Abel relation, we consider the following construction due to Bloch: For a field K, write

$$\mathcal{C}(K) := \left\langle [x] + [y] + \left[\frac{1-x}{1-xy}\right] + [1-xy] + \left[\frac{1-y}{1-xy}\right] \mid x, y \in K\backslash\{0,1\}, xy \neq 1 \right\rangle$$

for the subgroup of $\mathbb{Z}[K\backslash\{0,1\}]$ generated by things that look like the Spence-Abel relation. A theorem of Matsumoto implies that there is a surjection

$$K^* \wedge_{\mathbb{Z}} K^* \twoheadrightarrow K_2(K)$$

and the image of the map

$$\beta : \mathbb{Z}[K\backslash\{0,1\}] \to K^* \wedge_{\mathbb{Z}} K^*, \qquad [x] \mapsto x \wedge (1-x)$$

lands in the kernel. What's more, one can also check that $\beta(\mathcal{C}(F)) = 0$ so we have an exact sequence.

$$\frac{\mathbb{Z}[K\backslash\{0,1\}]}{\mathcal{C}(K)} \xrightarrow{\beta} K^* \wedge_{\mathbb{Z}} K^* \to K_2(K) \to 0\,.$$

The Bloch group $\mathcal{B}(K)$ of a field K is defined to be the kernel on the left, so we have an exact sequence

$$0 \to \mathcal{B}(K) \to \frac{\mathbb{Z}[K\backslash\{0,1\}]}{\mathcal{C}(K)} \xrightarrow{\beta} K^* \wedge_{\mathbb{Z}} K^* \to K_2(K) \to 0\,.$$

For our number field F, define a dilogarithmic map on $\mathcal{B}(F)$ by

$$D_F : \mathcal{B}(F) \to \mathbb{R}^{r_2}, \qquad [x] \mapsto (D(\sigma_{r_1+1}(x)), \ldots, D(\sigma_{r_1+r_2}(x)))\,.$$

It is well-defined because $D(\mathcal{C}(\mathbb{C})) = 0$.

Remark 2.1. Why do we only use the complex embeddings in the definition of D_F? Well, one can check that the Bloch-Wigner function satisfies $D(\overline{z}) = -D(z)$ (see the exercise sheet), so in particular D vanishes on $\mathbb{R}$.

This map looks kind of like the Dirichlet regulator from Lecture 1, but with dilogarithms instead of logarithms. Could it be related to the Borel regulator on $K_3(\mathcal{O}_F)$? It has the correct target $\mathbb{R}^{r_2}$ but the source is $\mathcal{B}(F)$ rather than $K_3(F)$. Well, Bloch constructed a homomorphism $\phi : \mathcal{B}(F) \to K_3(F)$ making the following diagram commute:

$$\begin{array}{ccc} K_3(F) & \xrightarrow{r_{\mathcal{B}o}} & \mathbb{R}^{r_2} \\ \uparrow{\scriptstyle \phi} & \nearrow_{D_F} & \\ \mathcal{B}(F) & & \end{array}$$

Suslin proved that the kernel and cokernel of ϕ are finite, so we have a commutative diagram

$$\begin{array}{ccc} K_3(\mathcal{O}_F)\otimes\mathbb{Q} & \xrightarrow{r_{\mathcal{B}o}} & \mathbb{R}^{r_2} \\ {\scriptstyle \phi\otimes\mathbb{Q}}\uparrow{\scriptstyle \cong} & \nearrow_{D_F} & \\ \mathcal{B}(F)\otimes\mathbb{Q} & & \end{array}$$

Borel's theorem says then that $r_{\mathcal{B}o}(K_3(\mathcal{O}_F)\otimes\mathbb{Q})) \cong D_F(\mathcal{B}(F)\otimes\mathbb{Q})$ is a $\mathbb{Q}$-structure on $\mathbb{R}^{r_2}$ and

$$\mathrm{vol}(r_{\mathcal{B}o}(K_3(\mathcal{O}_F)\otimes\mathbb{Q})) = \mathrm{vol}(D_F(\mathcal{B}(F)\otimes\mathbb{Q})) \sim_{\mathbb{Q}^*} |\mathrm{Disc}(F)|^{\frac{1}{2}}\pi^{2(d_2-d)}\zeta_F(2)\,.$$

This proves Zagier's conjecture for $\zeta_F(2)$.

The above suggests a strategy for proving Zagier's conjecture for $s = k \geq 3$ too. Indeed, Zagier defined higher analogues $\mathcal{B}_k(F)$ of the Bloch group (so $\mathcal{B}_2(F) = \mathcal{B}(F)$) and wrote down k-th polylogarithmic maps

$$P_{k,F} : \mathcal{B}_{k,F} \to \begin{cases} \mathbb{R}^{r_2} & \text{if } k \text{ is even} \\ \mathbb{R}^{r_1+r_2} & \text{if } k \text{ is odd} \end{cases}$$

(so $P_{2,F} = D_F$).

Beilinson-Deligne and de Jeu constructed a commutative diagram

$$\begin{array}{ccc} K_{2m-1}(\mathcal{O}_F)\otimes\mathbb{Q} & \xrightarrow{r_{\mathcal{B}o}} & \mathbb{R}^{d_{m-1}} \\ \uparrow{\scriptstyle \phi_m} & \nearrow_{P_{m,F}} & \\ \mathcal{B}_m(F) & & \end{array}$$

By Borel's theorem, to prove Zagier's conjecture it suffices to show that

$$\phi_m\otimes\mathbb{Q} : \mathcal{B}_m(F)\otimes\mathbb{Q} \to K_{2m-1}(\mathcal{O}_F)\otimes\mathbb{Q}$$

is surjective. Goncharov has shown that $\phi_3\otimes\mathbb{Q}$ is surjective thus confirming the "$\zeta_F(3)$ and trilogarithms" case of Zagier's conjecture.

Remarks 2.2.
(1) In principle, the above strategy allows for a computer to numerically "check" Zagier's conjecture for a given number field F and a given zeta value $\zeta_F(m)$. Indeed, Zagier supported his conjecture with a lot of computer evidence.
(2) Goncharov and Rudenko have recently proved the $s = 4$ case of Zagier's conjecture. I don't know how the proof goes.
(3) The injectivity of $\phi_m \otimes \mathbb{Q} : \mathcal{B}_m(F) \otimes \mathbb{Q} \to K_{2m-1}(\mathcal{O}_F) \otimes \mathbb{Q}$ is also an interesting and difficult problem.